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JACOBSON RINGS AND RINGS STRONGLY GRADED BY AN ABELIAN GROUP

BY

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ABSTRACT

Let R be ring strongly graded by an abelian group G of finite torsion-free rank. Let e be the identity of G, and R_e the component of degree e of R. Assume R_e is a Jacobson ring. We prove that graded subrings of R are again Jacobson rings if either R_e is a left Noetherian ring or R is a group ring. In particular we generalise Goldie and Michlers's result on Jacobson polycyclic group rings, and Gilmer's result on Jacobson commutative semigroup rings of finite torsion-free rank.

0. Introduction

If R is a commutative ring and S a commutative cancellative monoid of finite torsion-free rank, i.e. S is a submonoid of a commutative group of finite torsion-free rank, Gilmer [5] proved that the semigroup ring $R[S]$ is a Jacobson ring if and only if R is a Jacobson ring. This extends the commutative polynomial ring case $R[X]$ due to Goldman [8] and Krull [14]. In [18] Watters showed that the latter still holds if R is non-commutative.

In the case of skew polynomial rings $R[X, \phi]$, where ϕ is an automorphism of R, the question whether $R[X, \phi]$ is a Jacobson ring whenever R is a Jacobson ring was considered in [7], [16] and [17]. If R is left Noetherian the answer is positive; and in the other case counterexamples are known. As an application Goldie and Michler [7] obtained that a group ring of a polycyclic group over a left Noetherian Jacobson ring is a Jacobson ring.

In [4] Ferrero, Puczylowski and the author studied rings R strongly graded by a finitely generated abelian group. Particular interest is given to determine

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when R is a Jacobson ring. Positive results are obtained for example when R is left Noetherian or commutative. Again for R non-Noetherian counterexamples are given.

In this paper we investigate the same question for semigroup graded subrings of strongly G-graded rings, where G is an abelian group of finite torsion-free rank. Because of the previously mentioned results and examples we restrict ourselves to semigroup rings or to graded rings R with R_e left Noetherian.

1. Preliminaries

For the theory on semigroups and graded rings we refer to [2] and [15]. We briefly introduce some basic definitions and notations. All rings are associative and contain an identity element 1. If S is a group (or a monoid) then its identity element is denoted by e.

Let S be a semigroup. A ring R is called S-graded if there exist additive subgroups R_s of R, indexed by the elements of S, such that $R = \bigoplus_{s\in S} R_s$ and $R_sR_t \subseteq R_{st}$ for all $s, t \in S$. Any $r \in R$ can be written uniquely as a finite sum $\Sigma_{s\in S}\, r_s$, with $r_s \in R_s$. The summand r_s is the homogeneous component of degree s. The support of r is $\text{supp}(r) = \{s \in S \mid r_s \neq 0\}$. The set of all homogeneous elements of R is denoted $h(R)$, i.e. $h(R) = \bigcup_{s\in S} R_s$. If I is a (left) ideal of R, the largest homogeneous (left) ideal of R contained in I is denoted by $(I)_h$, i.e., $(I)_h = \bigoplus_{s\in S}(I \cap R_s)$. I is called homogeneous if $I = (I)_h$. A homogeneous ideal P of R is said to be gr-prime (graded-prime) if $I \cdot J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$ for all homogeneous ideals I and J of R. If R is $\mathbb{Z}$-graded, then P is gr-prime if and only if P is prime. Further, if T is a subset of S, then the set $\bigoplus_{t\in T} R_t$ is denoted $R_{[T]}$. In case T is a subsemigroup of S then $R_{[T]}$ is a T-graded ring. Note that $R_{[T]}$ is also a S-graded ring.

If X is a subset of a ring R, $r_R(X)$ (respectively $l_R(X)$) denotes the right (respectively left) annihilator of X in R. An ideal I of R is called right (respectively left) dense if $r_R(I) = 0$ (respectively $l_R(I) = 0$). If I is left and right dense then we simply say I is dense.

A G-graded ring R, where G is a group, is called weakly graded if $r_R(R_g) = l_R(R_g) = 0$ for all $g \in G$. If $R_gR_{g^{-1}} = R_e$, or equivalently $R_gR_h = R_{gh}$ for all $g, h \in G$, then R is called a strongly G-graded ring. Clearly every strongly graded ring is weakly graded.

Let R be a strongly G-graded ring, where G is an abelian group of finite torsion-free rank. Suppose S is a submonoid of G. The aim of this paper is to study when $R_{[S]}$ is a Jacobson ring in case either R_e is left Noetherian or $R = R_e[G]$, the group ring of G over R_e. Note that $(R_e[G])_{[S]} = R_e[S]$ the semigroup ring of the cancellative monoid S over R_e. The quotient group of S is denoted $\langle S \rangle$.

Recall that a ring R is called a Jacobson ring if the Jacobson radical $J(\bar{R})$ coincides with the prime radical $P(\bar{R})$, for every homomorphic image $\bar{R}$ of R. Equivalently, $J(R/P) = 0$ for every prime ideal P of R.

For our investigations we need a characterisation of prime ideals P of a weakly G-graded ring with $(P)_h = 0$, where G is an abelian group of torsion-free rank 0 or 1 (cf. Theorem 1.4). Such a characterisation is given in [4] in case R is strongly graded. However the proofs remain valid for weakly graded rings. Since [4] is as yet unpublished we state the results needed to prove the characterisation, and give a sketch of some of the proofs. But first we need the notion of symmetric graded-Martindale ring of quotients (cf. [13]).

Let G be a group and R a G-graded ring. For every $g \in G$ and right dense homogeneous ideal I of R we denote by $\mathrm{HOM}_R(I, R)_g$ the additive abelian group of all R-linear maps $f: I \to R$ which are graded homomophisms of degree g, i.e. $f(I_h) \subseteq R_{hg}$ for all $h \in G$. Let

$$\mathrm{HOM}_R(I, R) = \bigoplus_{g \in G} \mathrm{HOM}_R(I, R)_g.$$

The left graded-Martindale ring of quotients of R is

$$Q^l_{\text{gr-Mart}}(R) = \varinjlim \mathrm{HOM}_R(I, R),$$

i.e. the direct limit of the system

$$\{\mathrm{HOM}_R(I, R), \pi_{I,I'}: \mathrm{HOM}_R(I, R) \to \mathrm{HOM}_R(I', R),$$
$$I' \subseteq I, I' \text{ and } I \text{ right dense homogeneous ideals of } R\}.$$

The symmetric graded-Martindale ring of quotients of R is

$$Q^s_{\text{gr-Mart}}(R) = \{q \in Q^l_{\text{gr-Mart}}(R) \mid qK \subseteq R \text{ for some left dense homogeneous ideal } K \text{ of } R\}.$$

We recall (cf. [13]) the following properties of $Q = Q^s_{\text{gr-Mart}}(R)$:

(i) Q is a G-graded ring containing R as a graded subring, i.e. $R_g \subseteq Q_g$ for all $g \in G$;

(ii) for every $q \in Q$ there exists a right (respectively left) dense homogeneous ideal (respectively K) with $Iq \subseteq R$ and $qK \subseteq R$;

(iii) if $q \in Q$, I (respectively K) a right (respectively left) dense homogeneous ideal of R with $Iq = 0$ or $qK = 0$, then $q = 0$.

Throughout this section we denote by $Q(R)$, or simple Q, the symmetric graded-Martindale ring of quotients, and by $C(Q)$ the center of Q.

LEMMA 1.1. *Let R be a gr-prime G-graded ring, where G is an abelian group. The following are satisfied:*

(i) *$C(Q)$ is a graded field, i.e. $C(Q)$ is a graded ring of which every non-zero homogeneous element is invertible;*

(ii) *if $a_i, b_i \in h(Q)$ and $\Sigma\, a_i x b_i = 0$ for all $x \in h(R)$, then the a_i's are linearly dependent over $C(Q)$ and the b_i's are linearly dependent over $C(Q)$.*

PROOF. Similar to the ungraded versions (cf. [10]). □

Let R be a weakly G-graded ring, G an abelian group, and let I be a non-zero ideal of R. We say that a finite subset A of G satisfies (MS), the minimal support condition, if there exists $0 \neq a \in I$ such that $\mathrm{supp}(a) = A$ and A does not contain a proper subset T such that $T = \mathrm{supp}(b)$ for some $0 \neq b \in I$. Since R is weakly graded it follows that A satisfies (MS) if and only if $gA = \{ga \mid a \in A\}$ satisfies (MS).

LEMMA 1.2. *Let R be a gr-prime weakly G-graded ring, where G is an abelian group, and let I be a non-zero ideal of R. If $0 \neq a \in I$ satisfies* (MS), *then, for every $g \in \mathrm{supp}(a)$, $a = a_g c$ for some $c \in C(Q)$.*

PROOF. Let $0 \neq a \in I$ satisfy (MS). For every $r \in h(R)$ and $g \in \mathrm{supp}(a)$, $\mathrm{supp}(ara_g - a_g ra) \subseteq gh\,\mathrm{supp}(a)$ for some $h \in G$. Since $gh\,\mathrm{supp}(Q)$ satisfies (MS) it follows that $ara_g - a_g ra = 0$. Consequently, $a_s ra_g = a_g ra_s$ for every $s \in \mathrm{supp}(a)$. Lemma 1.1 implies $a_s = c(s)a_g$ for some $c(s) \in h(C(Q))$. Hence $a = a_g c$ for some $c \in C(Q)$. □

PROPOSITION 1.3. *Let R be gr-prime weakly G-graded ring and P a prime ideal of $Q = Q^s_{\text{gr-Mart}}(R)$ such that $(P)_h = 0$. Then $P = Q(P \cap C(Q))$.*

PROOF. Clearly Q is also weakly G-graded. Because of the previous lemma and the fact that $(P)_h = 0$, $a \in Q(P \cap C(Q))$ for every $a \in P$ which satisfies (MS). An induction argument on $\mathrm{supp}(b)$, $b \in P$, yields the result. □

THEOREM 1.4. *Let R be a weakly G-graded ring, where G is an abelian group of torsion-free rank* 0 *or* 1. *Let P be a non-zero ideal of R with $(P)_h = 0$. Then P is a prime ideal if and only if R is gr-prime and P is maximal with respect to $(P)_h = 0$.*

PROOF. Clearly P is prime if R is gr-prime and P is maximal with respect to $(P)_h = 0$.

The converse is proved using Proposition 1.3 and the fact that $C(Q)$ is integral over either a field (if G is torsion) or an infinite cyclic group algebra (if G has torsion-free rank 1). Hence all non-zero prime ideals of $C(Q)$ are maximal. □

We need one more application of Lemma 1.2 (cf. [3]).

LEMMA 1.5. *Let R be a prime weakly G-graded ring, where G is a torsion abelian group. If I is a non-zero ideal of R, then $(I)_h \neq 0$.*

PROOF. Under the assumptions, Q is a prime ring and thus $C(A)$ is an integral domain which is integral over the field $C(Q)_e$. Hence $C(Q)$ is a field.

Let $0 \neq a \in I$ satisfy (MS). Then, by Lemma 1.2, $a = a_g c$ for some $c \in C(Q)$. Since $c^{-1} \in C(Q)$, there exists a (dense) homogeneous ideal K of R with $c^{-1}K \subseteq R$. Consequently,

$$a_g K = a_g c(c^{-1}K) = ac^{-1}K \subseteq I.$$

So $(I)_h \neq 0$. □

2. Positively graded rings

Let R be a G-graded ring, G an abelian group. An ideal I of R_e is said to be G-invariant if $R_g I R_{g^{-1}} \subseteq I_e$ for all $g \in G$. In case R is strongly graded it follows that $R_g I R_{g^{-1}} = I$, and thus $RI = IR$. The latter means that the ideal I of R_e extends to an ideal of R.

If R is a $\mathbf{Z}$-graded ring, $\mathbf{Z}$ the additive group of integers, then we denote by R_+ (respectively R^+) the set $R_{[\mathbf{N}]}$ (respectively $R_{[\mathbf{N}_0]}$), where $\mathbf{N}$ is the set of non-negative integers and $\mathbf{N}_0 = \mathbf{N} \setminus \{0\}$. R_+ is a $\mathbf{N}$-graded ring (called the positive part of R) of which R^+ is an ideal.

PROPOSITION 2.1. *Let R be a strongly $\mathbf{Z}$-graded ring and P a non-zero ideal of R_+ with $(P)_h = 0$. Assume that either ideals of R_0 extend to ideals of R, or that R_0 is left Noetherian. Then P is a prime ideal of R_+ if and only if R_+ is prime and P is maximal with respect to $P \cap R_0 = 0$.*

PROOF. Clearly P is prime if R_+ is prime and P is maximal with respect to $P \cap R_0 = 0$. For the converse, assume P is prime with $(P)_h = 0$. Obviously R_+ is then gr-prime, and thus prime (cf. [15]). Because $(P)_h = 0$, $R_n \not\subseteq P$ for all $n \in \mathbf{N}$. It follows that $RPR \cap R_+ = P$. Because $I = R(I \cap R_+)$ for every ideal I

of R, it follows easily that $P' = RPR$ is a prime ideal of R with $(P')_h = 0$. Because of Theorem 1.4, P' is maximal with respect to $(P')_h = 0$.

If P is not maximal with respect to $P \cap R_0 = 0$, then let $P \subsetneq I$ be an ideal of R_+ maximal with respect to $I \cap R_0 = 0$. In case all ideals of R_0 extend to ideals of R, it follows that R_0 is prime (because R_+ is prime); hence I is a prime ideal of R_+. Further $R_1 \not\subseteq I$. For if not then $I = R^+$ and thus $P \subseteq R^+$. However this is impossible as $RPR \cap R_+ = P$. Hence $R_n \not\subseteq I$ for all n, and thus $RIR \cap R_+ = I$. We now prove that the same holds if R_e is left Noetherian. In this case $\mathscr{C}(0)$, the set of regular elements of R_0, is a left Ore set. Let $I \subseteq M$, where M is an ideal of R_+ maximal with respect to $M \cap \mathscr{C}(0) = 0$. Clearly M is a prime ideal of R_+. For an ideal N of R_+ we denote by

$$\gamma(N) = \{r_0 \in R_0 \mid r_0 \text{ is the 0-degree term of some } r \in N\}.$$

It follows that $\gamma(N)$ is an ideal of R_0. Now since $RPR \cap R_+ = P$, we obtain that $0 \neq \gamma(P)$ is an invariant ideal of R_0. Because R_+ is prime this implies $\gamma(P)$ is dense in R_0. Further $\gamma(P) \subseteq \gamma(M)$. Hence $\gamma(M)$ is dense in R_0. So $\mathscr{C}(0) \cap \gamma(M) \neq 0$. It then follows that $R_1 \not\subseteq M$. For if $R_1 \subseteq M$ then $M = \gamma(M) + R^+$, in particular $M \cap \mathscr{C}(0) \neq 0$, a contradiction. Since M is prime and $R_1 \not\subseteq M$, $RMR \cap R_+ = M$. Hence $M \cap R_0$ is an invariant ideal of R_+, and $M \cap R_0 \neq 0$ if and only if $(M)_h \neq 0$. Again since R_+ is prime, $M \cap R_0 \neq 0$ if and only if $(M \cap R_0) \cap \mathscr{C}(0) \neq 0$. By definition of M the latter is impossible. Thus $M \cap R_0 = 0$. Hence $M = I$, $RIR \cap R_+ = I$, $(I)_h = 0$.

So in both cases, $P' \subsetneq RIR$ and $(RIR)_h = 0$, a contradiction. □

THEOREM 2.2. *Let R be a strongly $\mathbb{Z}$-graded ring such that either ideals of R_0 extend to ideals of R, or R_0 is left Noetherian. Assume that $J(R_+/P) = 0$ and $P \cap R_0$ is a semiprime ideal of R_0, for every prime homogeneous ideal P of R_+. Then R_0 is a Jacobson ring if and only if R_+ is a Jacobson ring.*

PROOF. Since R_0 is a homomorphic image of R_+, R_0 is a Jacobson ring if R_+ is a Jacobson ring. For the converse, we have to prove that $J(R/P) = 0$ for every prime ideal P of R_+. Because of the assumptions we may assume $(P)_h \subsetneq P$ and $R_1 \not\subseteq P$. Since $(P)_h$ is also a prime ideal of R_+ it follows that $R(P)_hR \cap R_+ = (P)_h$. Hence $T = R/R(P)_hR$ is a strongly $\mathbb{Z}$-graded ring with $T_+ = R_+/(P)_h$, $T_0 = R_0/P \cap R_0$ and $P/(P)_h$ does not contain homogeneous elements. So, replacing R by T, we may assume P is a non-zero prime ideal of R_+, with $(P)_h = 0$; in particular $P \cap R_0 = 0$. Hence (by the assumptions) R_0 is semiprime and R_+ is a prime ring.

For $0 \neq r_0 + \cdots + r_n = r \in R_+$ we denote r_n, the leading term of r, by $\mathrm{ld}(r)$

and n by $\delta(r)$. Let $n = \min\{\delta(r) \mid 0 \neq r \in P\}$. Note that $n > 0$ since $P \cap R_0 = 0$. Let

$$M(P) = \{r_n \in R_n \mid r_n = \mathrm{ld}(r),\ n = \delta(r) \text{ for some } 0 \neq r \in P\}.$$

Denote $\tau(P) = M(P)R_{-n}$. Clearly $\tau(P)$ is a $\mathbf{Z}$-invariant ideal of R_0.

We claim that if $q \in \tau(P)R_+$ with $\delta(q) \geqq n$, then $\delta(q-p) < \delta(q)$ for some $p \in P$. Indeed, let $q = q_0 + \cdots + q_k$ with $k = \delta(q) \geqq n$. Then $q_k \in M(P)R_{-n+k}$, say $q_k = \Sigma_{i=1}^{l} b_i x_i$ where $b_i \in M(P)$ and $x_i \in R_{-n+k}$ for all $1 \leqq i \leqq l$. Hence there exist $a_i \in P$ with $\delta(a_i) = n$ and $\mathrm{ld}(a_i) = b_i$. Consequently $p = \Sigma\, a_i x_i \in P$ and $\delta(q-p) < k$.

Assume $J(R_+/P) = I/P \neq 0$, where I is an ideal of R_+ containing P. Because of Proposition 2.1, $I \cap R_0 \neq 0$. We claim that $\tau(P)(I \cap R_0) \subseteq J(R_0)$. So let $r_0 \in \tau(P)(I \cap R_0)$. Then, for some $q' \in R_+$, $r_0 + q' + r_0 q' \in P$. It follows that $q' \in -r_0(1+q') + P$. Let $q = -r_0(1+q')$ then $r_0 + q + r_0 q \in P$ and $q \in \tau(P)(I \cap R_0)R_+$. Using the previous claim and an induction argument, we may assume $\delta(q) < n$. Consequently $\delta(r_0 + q + r_0 q) < n$. The minimality of n implies $r_0 + q + r_0 q = 0$. Hence, taking the component of degree 0, $r_0 + q_0 + r_0 q_0 = 0$. So r_0 is right quasi-invertible. This proves the claim. Since R_0 is a semiprime Jacobson ring we obtain $\tau(P)R_+(I \cap R_0) = R_+\tau(P)(I \cap R_0) = 0$. This contradicts R_+ being prime. This finishes the proof. □

Corollary 2.3 (Watters [18]). *For any ring R, R is a Jacobson ring if and only if the polynomial ring $R[X]$ is a Jacobson ring.*

Proof. Let R be Jacobson ring. Let $T = R[X, X^{-1}]$ by the cyclic group ring over R. Then T is strongly $\mathbf{Z}$-graded, $T_+ = R[X]$, $T_0 = R$ and ideals of R extend to ideals of T. To prove that $R[X]$ is a Jacobson ring we only have to verify that the assumptionns of Theorem 2.2 are satisfied. For this let P be a homogeneous prime ideal of $R[X]$. Obviously $P \cap R$ is a prime ideal of R. Further if $X \in P$ then $R[X]/P \cong R/P \cap R$. Hence (R is a Jacobson ring) $J(R[X]/P) = 0$. If $X \notin P$, then $P = p[X]$ with $p = P \cap R$. Hence $R[X]/P = (R/p)[X]$ and $J(R/p) = 0$. It is then well-known (cf. [1]) that $J(R[X]/P) = 0$.

Corollary 2.4. *Let R be a strongly $\mathbf{Z}$-graded ring with R_0 left Noetherian. Then, R_+ is a Jacobson ring if and only if R_0 is a Jacobson ring.*

Proof. Assume R_0 is a Jacobson ring. We verify that the assumptions of Theorem 2.2 are satisfied. Let P be a homogeneous prime ideal of R_+. If $R_1 \subseteq P$ then it is clear that $P \cap R_0$ is a prime ideal of R_0 and

$$J(R_+/P) \cong J(R_0/P \cap R_0) = 0.$$

If $R_1 \not\subseteq P$ then $RPR \cap R_+ = P$. As before, $T = R/RPR$ is a strongly $\mathbf{Z}$-graded ring, $T_+ = R_+/P$ and $T_0 = R_0/P \cap R_0$ is left Noetherian. It follows from [11] that $P \cap R_0$ is semiprime and $J(R_+/P_+) = 0$. □

An immediate application of the last corollary is Goldie and Michler's result [7]. A skew polynomial ring $R[X, \phi]$, ϕ an automorphism of R, is a left Noetherian Jacobson ring if and only if R is a left Noetherian Jacobson ring.

Without the assumption that R is strongly graded Corollary 2.4 is not valid in general. We give an example.

EXAMPLE. Let k be a field and $T = k[[X]]$ the formal power series over k. Let $R = \bigoplus_{n \in Z} R_h$ be the following $\mathbf{Z}$-graded subring of the polynomial ring $T[Y]$. For $n < 0$, $R_n = 0$; for $n > 0$, $R_n = TY^n$ and $R_0 = k$. So $R = R_+$ is the set of polynomials $f(Y) \in T(Y)$ with constant term $f(0) \in k$. Clearly $P = \{f(Y) \in T[Y] \mid f(1) = 0\}$ is a prime ideal of $T[Y]$. Hence $p = R \cap P$ is a prime ideal of R and $R/p \cong T[Y]/P \cong T$. Since $J(T) \neq 0$, R is not a Jacobson ring. As each R_n is a k-vector space, and because the dimensions of R_0 and R_1 are different, it is clear that R_+ cannot be the positive part of a strongly $\mathbf{Z}$-graded ring.

3. Abelian group graded rings

To prove our main results we need information on homomorphic images of G-graded rings, that is G-systems, where G is an abelian group. Recall (cf. [9]), a ring T is said to be a G-system if $T = \Sigma_{g \in G} T_g$ (not necessarily a direct sum), where T_g is an additive subgroup of T and $T_g T_h \subseteq T_{gh}$ for all $g, h \in G$.

LEMMA 3.1 (Grzeszczuk [9]). *Let T be a G-system, G a torsion abelian group. Then $J(T) \cap T_e \subseteq J(T_e)$.*

PROOF. Let $a \in J(T) \cap T_e$. Then $(1 + a)b = 1$ for some $b \in T$. Since $b \in T_{[H]} = \Sigma_{h \in H} T_h$ for some finite subgroup H of G, $(1 + a)$ is invertible in the H-system $T_{[H]}$. Hence (cf. [9]) $1 + a$ is invertible in T_e. Therefore $J(T) \cap T_e$ is a quasi-regular ideal of T_e and the result follows. □

LEMMA 3.2. *Let G be a torsion abelian group and R a weakly G-graded ring with $RR_g = R_gR$ for all $g \in G$. Let P be a prime ideal of R such that $R_g \not\subseteq P$ for all $g \in G$. If $J(R_e/P \cap R_e) = 0$ then $J(R/P) = 0$.*

PROOF. Let $\bar{R} = R/(P)_h$. Clearly $\bar{R}$ is a gr-prime G-graded ring and, because of the assumptions, $\bar{R}_g \neq 0$ for all $g \in G$. Hence $\bar{R}$ is weakly graded.

So we may assume $R = \bar{R}$ and $(P)_h = 0$, and we have to prove $J(R/P) = 0$ in case $J(R_e) = 0$.

If $P = 0$, then R is a prime ring. Hence because of Lemma 1.5, $J(R) \neq 0$ implies $(J(R))_h \neq 0$. Since R is weakly graded and because of Lemma 3.1, the latter implies $0 \neq J(R) \cap R_e \subseteq J(R_e)$.

Assume now $0 \neq P$ and $J(R/P) = I/P \neq 0$, where I is an ideal containing P. Because of Theorem 1.4, $R_g \cap I \neq 0$ for some $g \in G$. Hence

$$0 \neq RR_{g^{-1}}(R_g \cap I) = R_{g^{-1}}R(R_g \cap I) \not\subseteq P.$$

Thus $I_e = I \cap R_e \not\subseteq P$. Now $T = R/P = \Sigma_{g \in G} T_g$, where $T_g = R_g + P$, is a G-system. So by Lemma 3.1, $J(T) \cap T_e \subseteq J(T_e)$. This means $0 \neq I_e \subseteq J(R_e)$. This finishes the proof. □

LEMMA 3.3. *Let R be a strongly G-graded ring, G an abelian group with submonoid S. Let F be a free submonoid of S of finite rank, and assume R_e is left Noetherian. If P is a prime ideal of $R_{[S]}$ with $R_s \not\subseteq P$ for every $s \in S$, then $P \cap R_{[F]}$ is a semiprime ideal of $R_{[F]}$.*

PROOF. It is well-known (cf. [15]) that $R_{[F]}$ is left Noetherian under the assumptions. We have to prove $N/(R_{[F]} \cap P) = 0$, where $N = \bigcap M$, the intersection of all prime ideals M of $R_{[F]}$ containing $P \cap R_{[F]}$. Clearly for such a prime ideal M, $R_sMR_{s^{-1}}$ and $R_{s^{-1}}MR_s$ are again prime ideals of $R_{[F]}$ for any $s \in S$. Further, since $(R_s(P \cap R_{[F]})R_{s-1})R_s \subseteq P$ and since $R_s \not\subseteq P$, it follows that

$$R_s(P \cap R_{[F]})R_{s^{-1}} \subseteq P \cap R_{[F]}.$$

Similarly

$$R_{s^{-1}}(P \cap R_{[F]})R_s \subseteq P \cap R_{[F]}.$$

Hence

$$R_s(P \cap R_{[F]})R_{s^{-1}} = P \cap R_{[F]} \qquad \text{for every } s \in S,$$

and $P \cap R_{[F]} \subseteq R_sMR_{s^{-1}}$. Consequently $R_sNR_{s^{-1}} = N$ for all $s \in S$. Thus $R_{[S]}N = NR_{[S]}$. Since P is a prime ideal of $R_{[S]}$ it follows that $R_{[S]}N$ is not nilpotent modulo P. Hence N is not nilpotent modulo $R_{[F]} \cap P$. Since $R_{[F]}$ is left Noetherian the result follows. □

LEMMA 3.4. *Let S be an abelian semigroup with subsemigroup T. If P is a prime ideal of the semigroup ring $R[S]$, then $P \cap R[T]$ is a prime ideal of $R[T]$.*

PROOF. This is obvious since ideals of $R[T]$ extend to ideals of $R[S]$. □

LEMMA 3.5. *Let R be a strongly G-graded ring, G an abelian group, and S a* *submonoid of G. Let P be a prime ideal of $R_{[S]}$ and denote $T = S \setminus A$ where $A = \{s \in S \mid R_s \subseteq P\}$. Then the following are satisfied:*

(i) *T is a submonoid of G;*

(ii) *$R_{[S]}/(P)_h = \bar{R}_{[T]}$, for some strongly $\langle T \rangle$-graded ring $\bar{R}$ with $\bar{R}_e \cong R_e/P \cap R_e$;*

(iii) *$P/(P)_h = \bar{P}$ for some prime ideal $\bar{P}$ of $\bar{R}_{[T]}$ with $(\bar{P})_h = 0$.*

PROOF. Because P is a prime ideal it follows that A is a prime ideal of S, possibly empty. Hence T is a submonoid of S, and thus of G. Clearly

$$R_{[S]}/(P)_h \cong R_{[T]}/R_{[T]} \cap (P)_h.$$

Further let $R_{[\langle T \rangle]}(R_{[T]} \cap (P)_h)R_{[\langle T \rangle]} = P'$, then $P' \cap R_{[T]} = R_{[T]} \cap (P)_h$. Hence $\bar{R} = R_{[\langle T \rangle]}/P'$ is a strongly $\langle T \rangle$-graded ring with

$$\bar{R}_{[T]} \cong R_{[T]}/R_{[T]} \cap (P)_h \quad \text{and} \quad \bar{R}_e \cong R_e/R_e \cap P.$$

Moreover, $\bar{P} = (R_{[T]} \cap P)/R_{[T]} \cap (P)_h \cong P/(P)_h$ is a prime ideal of $\bar{R}_{[T]}$ with $(\bar{P})_h = 0$. □

We are now able to prove the main theorem.

THEOREM 3.6. *Let S be a submonoid of an abelian group G of finite torsion-free rank n. Let R be a strongly G-graded ring. If R_e is a Jacobson ring then $R_{[S]}$ is a Jacobson ring in each of the following cases:*

(i) *R_e is left Noetherian,*

(ii) *$R = R_e[G]$.*

PROOF. We prove the result by induction on n. If $n = 0$ then S is contained in a torsion group. So S itself is a torsion group. Hence for any prime ideal P of $R_{[S]}$, $R_s \not\subseteq P$ for all $s \in S$. So the result follows from Lemma 3.2, Lemma 3.3 and Lemma 3.4.

So assume $n > 0$. Let P be a prime ideal of $R_{[S]}$. We have to prove $J(R_{[S]}/P) = 0$. Because of Lemma 3.5 and the induction hypothesis, we may assume $(P)_h = 0$ (note that $R_{[S]}/(P)_h$ is again a semigroup ring in case $R_{[S]} = R_e[S]$) and $\langle S \rangle = G$ is of torsion-free rank n. In particular $R_s \not\subseteq P$ for all $s \in S$. Let F be a free submonoid of S such that $G/\langle F \rangle$ is a torsion group. We define the following congruence relation $\sim$ on S: for $s, t \in S$, we say $s \sim t$ if $as = bt$ for some $a, b \in F$. Then the quotient monoid $S/\sim$ is a submonoid of $G/\langle F \rangle$. Since the latter is a torsion group the same is true for $S/\sim$. So we can consider, in a natural way, $R_{[S]}$ as a weakly $S/\sim$-graded ring, with component of degree e $R_{[S \cap \langle F \rangle]}$. Hence, by Lemma 3.2, it is sufficient to prove that

$$J(R_{[S\cap\langle F\rangle]}/P \cap R_{[S\cap\langle F\rangle]}) = 0.$$

Since $P \cap R_{[F]}$ is a semiprime ideal of $R_{[F]}$ (Lemma 3.3 and Lemma 3.4), $P \cap R_{[S\cap\langle F\rangle]}$ is a semiprime ideal of $R_{[S\cap\langle F\rangle]}$. So it is sufficient to prove that $R_{[S\cap\langle F\rangle]}$ is a Jacobson ring. Moreover, since $\langle S \cap \langle F\rangle\rangle = \langle F\rangle$, we may assume $\langle S\rangle$ is a free group of rank n.

So we have reduced the problem to the following situation. R is a strongly G-graded ring, S a submonoid of G and F a free monoid, with generators $f_1, \ldots, f_n$, contained in S such that $\langle S\rangle = G = \langle F\rangle$. Further P is a prime ideal of $R_{[S]}$ with $(P)_h = 0$. Let $r \in R_{[S]}$ be such that $r + P \in J(R_{[S]}/P)$. Since $\langle S\rangle = \langle F\rangle$, $rR_t \subseteq R_{[F]}$ for some $t \in F$. Let $W = rR_tR_{f_1}\cdots R_{f_n}$.

We claim that $W + (R_{[F]} \cap P) \subseteq J(R_{[F]}/R_{[F]} \cap P)$. We prove that $W \subseteq L$ for every maximal left ideal L of $R_{[F]}$ with $R_{[F]} \cap P \subseteq L$. Firstly we consider the case where $R_f \subseteq L$ for some $f \in F$, say $f = f_1^{k_1}\cdots f_n^{k_n}$, $k_i \in \mathbb{N}$. Obviously we may assume all $k_i \neq 0$. Then

$$(R_{f_1}\cdots R_{f_n})^k \subseteq L \qquad \text{for } k = k_1 + \cdots + k_n.$$

Let k be the smallest number for which $(R_{f_1}\cdots R_{f_n})^k \subseteq L$. We claim $k = 1$. If not, then $R_{[F]}(R_{f_1}\cdots R_{f_n})^{k-1} + L = R_{[F]}$. Hence

$$(R_{f_1}\cdots R_{f_n})R_{[F]}(R_{f_1}\cdots R_{f_n})^{k-1} + (R_{f_1}\cdots R_{f_n})L = R_{f_1}\cdots R_{f_n}R_{[F]},$$

and thus $R_{f_1}\cdots R_{f_n}R_{[F]} \subseteq L$, a contradiction. We obtain $W = rR_tR_{f_1}\cdots R_{f_n} \subseteq L$. Secondly assume $R_f \not\subseteq L$ for all $f \in F$. In this case we claim $(R_{[S]}L + P) \cap R_{[F]} = L$. If not, then, because L is a maximal left ideal of $R_{[F]}$,

$$(R_{[S]}L + P) \cap R_{[F]} = R_{[F]}.$$

But then for some $f \in F$, $R_f \subseteq L + (P \cap R_{[F]}) \subseteq L$, a contradiction. Let then M be a left ideal of $R_{[S]}$ containing P and maximal with respect to $M \cap R_{[F]} = L$. It follows that M is a maximal left ideal of $R_{[S]}$ containing P. Hence $W \subseteq M$, and therefore $W \subseteq M \cap R_{[F]} = L$.

Now, because of Lemma 3.3, $P \cap R_{[F]}$ is a semiprime ideal of $R_{[F]}$. Further, because of Corollary 2.3 and Corollary 2.4, $R_{[F]}$ is a Jacobson ring. Hence $W + (R_{[F]} \cap P) \subseteq J(R_{[F]}/R_{[F]} \cap P) = 0$. Thus $rR_tR_{f_1}\cdots R_{f_n} \subseteq P$, and thus because $(P)_h = 0$, $r \in P$. This proves $J(R_{[S]}/P) = 0$. □

COROLLARY 3.7. *Let R be a Jacobson ring.*

(1) (Gilmer [6] for R commutative) *Let T be a ring and $a_1, \ldots, a_n$ central elements of T. If $T = R[a_1, \ldots, a_n]$, i.e. T is a finitely generated ring*

extension of R, then any intermediate ring A generated over R by monomials $a_1^{e_1}\cdots a_n^{e_n}$ in the set $\{a_i\}_{i=1}^n$ is a Jacobson ring.

(2) (Goldie and Michler [7]) *The group ring $R[G]$, G a polycyclic group, is a left Noetherian Jacobson ring if and only if the same is true for R.*

PROOF. Since A is a homomorphic image of a semigroup ring $R[S]$ where S is a submonoid of a free semigroup of finite rank, the corollary is an immediate consequence of Theorem 3.6. □

Added in proof. We thank the referee for pointing out that A. Bell in *Localization and ideal theory in Noetherian group graded rings*, J. Algebra **105** (1987), 76–115, has proved that a right Noetherian crossed product $R * G$ is a Jacobson ring if R is a Jacobson ring and G is a polycyclic-by-finite group.

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